Spectral Cross Correlations of Magnetic Edge States

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We observe strong, nontrivial cross correlations between the edge states found in the interior and the exterior of magnetic quantum billiards. Our analysis is based on a novel definition of the edge state spectral density which is rigorous, practical, and semiclassically accessible.

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One of the main goals in the field of "quantum chaos" is to link the autocorrelations found in a quantum spectrum to the periodic orbits of the classical problem [1]. Here, we extend this study and investigate whether *cross* correlations exist between quantum systems which are different but related by their classical dynamics. We develop this idea for magnetic quantum billiards [2–4] which often serve to model semiconductor quantum dots [5].

Magnetic billiards consist of a charged particle moving ballistically in a compact domain in the plane subject to a homogeneous magnetic field. The quantum wave function is required to vanish at the billiard boundary while the impinging classical particle is reflected specularly [6–8]. The boundary defines also a complementary problem — an antidot — where the particle is confined to the exterior and is scattered at the billiard boundary. Although the exterior domain is unbounded, its spectrum is *discrete* [9]. Is it possible to relate the energy levels of the dot to those of the antidot? We show that there exists an intimate, nontrivial connection between the spectra of the interior and the exterior problem. It is the quantum manifestation of a duality in the classical dynamics.

The classical *interior-exterior duality* is illustrated in Fig. 1(a): Since a (periodic) orbit consists of arcs of constant curvature, one can construct a *dual* orbit in the complementary domain by completing the arcs to circles. Any skipping trajectory meets with a dual one under rather general conditions — if every circle of cyclotron radius ρ intersects the boundary at most twice. Pairs of dual periodic orbits have the same stability and their actions add up to an integer multiple of the action of a cyclotron orbit. On semiclassical grounds one may therefore expect the correlation between the interior and the exterior motion to carry over to the quantum spectrum. This is also corroborated by the existence of pairs of interior and exterior quantum eigenstates which match up well, cf. Fig. 1(b), although their energies differ.

The semiclassical analysis is complicated by the fact that in the exterior each Landau level is an accumulation point for an infinite series of energies. The respective states — the *bulk states* — correspond to unperturbed cyclotron motion. Also in the interior one may find (a finite number of) bulk states if ρ permits complete cyclotron orbits to fit into the domain. The eigenfunctions which correspond to the skipping trajectories, on the other hand, are called *edge states.* Clearly, a possible correlation is to be expected only between these nontrivial exterior and interior states.

Although the notion of edge states is intuitively clear and often used (e.g., in the context of the quantum Hall effect [10]), we are not aware of a general quantitative definition in the literature (however, see [11]). Therefore,

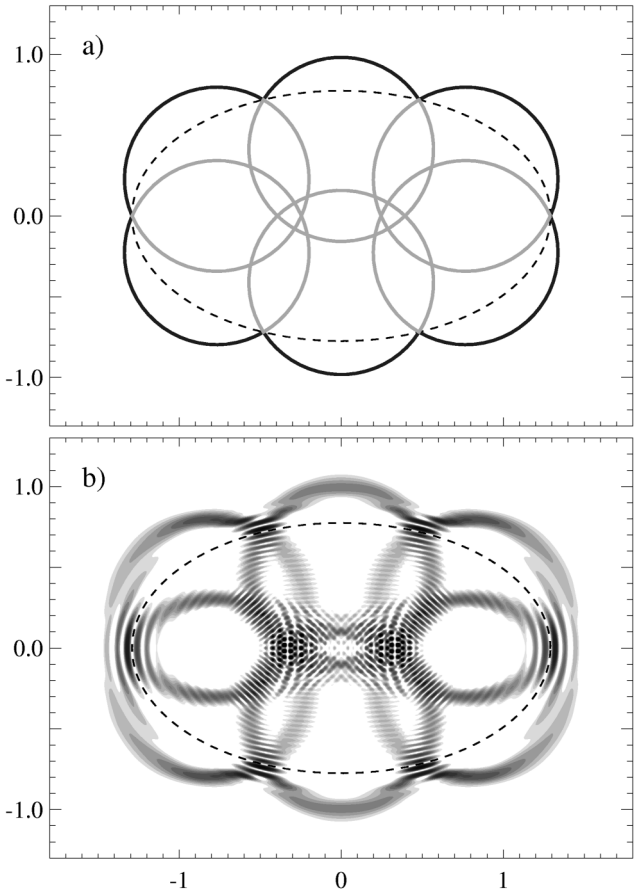

FIG. 1. (a) A pair of dual periodic trajectories and (b) a pair of correlated eigenfunctions found in the interior and the exterior ellipse billiard (superimposed). The billiard boundary is indicated by a dashed line.

the purpose of this Letter is twofold. First, we propose a definition for the spectral density of edge states which provides a meaningful characterization and applies in the quantum and in the semiclassical regime. Only with this can we then establish the existence of a pairwise relation between the edge states of the interior and the exterior.

A definition for edge states should take into account that a clear separation into edge and bulk occurs only in the semiclassical limit $b \to 0$. Here, we express the quantum scale in terms of the magnetic length $b = \sqrt{2\hbar/m\omega_c}$ (with m the mass and ω_c the cyclotron frequency). At finite values of b the states may be of an intermediate type. We propose to quantify the transition by attributing a weight $w_i > 0$ to each eigenstate ψ_i (of energy ν_i) which gives a measure of the degree to which ψ_i has the character of an edge state. The spectral density of edge states in either the interior or the exterior is then defined as

$$d_{\mathrm{edge}}(\nu) = \sum_{i=1}^{\infty} w_i \delta(\nu - \nu_i). \tag{1}$$

Here, we scale the energy E by the spacing between Landau levels, $\nu = E/(\hbar\omega_c) = \rho^2/b^2$. Any reasonable definition of the weights w_i must suppress the bulk states by exponentially small values, such that the mean edge density $\overline{d}_{\mathrm{edge}}$ is well defined in the exterior and equal to the interior one, to leading order. In the semiclassical limit it should match our notion of edge states admitting a trace formula which involves only the skipping trajectories. Moreover, we demand $\overline{d}_{\mathrm{edge}}$ to coincide with the unweighted interior mean density if the cyclotron radius is large enough to prevent bulk states.

To motivate our definition of the weights w_i, consider the *scaled* magnetization $\mathcal{M}$ of the interior billiard, a sum over the scaled magnetic moments

$$\mathcal{M}(\nu; b) = \sum_{\nu_i \le \nu} \frac{\langle \psi_i | \mathbf{r} \times \mathbf{v} | \psi_i \rangle}{\omega_c b^2} = \int_0^{\nu} m(\nu'; b)\, d\nu'. \tag{2}$$

The scaled magnetization density $m(\nu)$ can be expressed by the derivatives of the spectral counting function $N(\nu; b) = \sum_i \Theta[\nu - \nu_i(b)]$ with respect to b^2 and ν,

$$\begin{aligned} m(\nu) &= -b^2 \frac{\partial N}{\partial b^2} - \nu \frac{\partial N}{\partial \nu} \\ &= \sum_i \left(b^2 \frac{d\nu_i}{db^2} - \nu_i \right) \delta(\nu - \nu_i). \end{aligned} \tag{3}$$

This is verified by replacing the energies ν_i in (3) by the expectation values of the Hamiltonian. The scaled magnetization density may be obtained from the conventional one by multiplication with the field strength B. It exhibits a natural partitioning into a bulk part and an edge part since it complies with the scaling properties of the system: The scaled magnetic moment of a Landau state is $-\nu$. Hence, the second part of (3),

$$m_{\mathrm{bulk}}(\nu) = -\nu \frac{\partial N}{\partial \nu} = \sum_i (-\nu_i) \delta(\nu - \nu_i), \tag{4}$$

attributes the full diamagnetic response of a Landau state to each state ψ_i. We call it the bulk magnetization density. It follows that the remaining part, the *edge magnetization* density

$$m_{\mathrm{edge}}(\nu) = -b^2 \frac{\partial N}{\partial b^2} = \sum_{i=1}^{\infty} b^2 \frac{d\nu_i}{db^2} \delta(\nu - \nu_i) \tag{5}$$

assigns the positive excess magnetic moments induced by the presence of a billiard boundary. Its mean value

$$\overline{m}_{\mathrm{edge}}(\nu) = \frac{\mathcal{A}}{b^2 \pi} \nu - \frac{1}{2} \frac{\mathcal{L}}{2\pi b} \nu^{1/2} = -\overline{m}_{\mathrm{bulk}}(\nu) \tag{6}$$

follows from the mean number of states in a magnetic billiard with area $\mathcal{A}$ and circumference $\mathcal{L}$ [12],

$$\overline{N}(\nu; b) = \frac{\mathcal{A}}{b^2 \pi} \nu - \frac{\mathcal{L}}{2\pi b} \nu^{1/2} + \frac{1}{6}. \tag{7}$$

Note that $\overline{m}_{\mathrm{edge}}$ cancels the mean bulk magnetization density exactly: There is no orbital magnetism apart from the quantum fluctuations. Hence, m_{edge} characterizes those few (edge) states which carry a finite current along the boundary, balancing the bulk magnetization due to their large positive magnetic moments.

The edge magnetization is well defined in the exterior as well. There, it is negative with the mean like (6) except for a minus sign in front of the area term. This suggests to define the edge state density as $d_{\mathrm{edge}}(\nu) = \pm m_{\mathrm{edge}}(\nu)/\nu$, with the lower sign for the exterior problem. The corresponding weights

$$w_i = \pm \frac{b^2}{\nu_i} \frac{d\nu_i}{db^2} = \pm \frac{1}{\nu} \left(\frac{\langle \psi_i | \mathbf{r} \times \mathbf{v} | \psi_i \rangle}{\omega_c b^2} + \nu \right) > 0 \tag{8}$$

are easily obtained as the derivative of the eigenenergies taken at fixed ρ. This definition satisfies the conditions formulated above. In particular, the weights are exponentially small for bulk states since the Landau energies $\nu = N + \frac{1}{2}$, $N \in \mathbb{N}_0$ are independent of b [13].

The semiclassical edge state density is derived by inserting the trace formula for $N(\nu)$ [14,15] into (5). One obtains a sum over all skipping periodic orbits for the fluctuating part $\tilde{d}_{\mathrm{edge}} = d_{\mathrm{edge}} - \overline{d}_{\mathrm{edge}}$. It differs from the semiclassical expression of the unweighted spectral density only by a factor

$$w_\gamma = \frac{2\mathcal{A}_\gamma \pm \rho \mathcal{L}_\gamma}{\rho \mathcal{L}_\gamma} \tag{9}$$

attributed individually to each periodic orbit contribution. This *classical weight* is determined by the area $\mathcal{A}_\gamma$ enclosed by the trajectory γ and by its length $\mathcal{L}_\gamma$. The weights approach zero as the skipping orbits are further detached from the boundary. Hence, the classical weights (9) smoothly suppress the bulk contributions to the semiclassical spectral density.

Figure 2 shows (a) the quantum weights w_i against their energies ν_i [9] and (b) the phase space distribution of the classical weights [for the interior ellipse billiard with area $\mathcal{A} = \pi$ and eccentricity 0.8 at $b = 0.1$; the shade in (b) gives the probability measure for finding a trajectory with weight w_γ]. In Fig. 2(a) one observes how the w_i distinguish the relevant edge states from the bulk. Characterized by vanishingly small w_i, the bulk states accumulate at the Landau levels, $\nu = N + \frac{1}{2}$, with sequences of transitional states connecting to the large edge weights. The latter distribute in structures reproduced by the classical weights (9) (which are due to bifurcating regular islands in phase space). In the exterior, the segregation into edge and bulk is even more distinct (not shown) [16].

In order to unravel the relation between the interior and the exterior spectra we consider the cross correlator

$$C(\nu_0) = \int\int \tilde{d}^{(\mathrm{int})}_{\mathrm{edge}}(\nu; b^2)\tilde{d}^{(\mathrm{ext})}_{\mathrm{edge}}(\nu; b^2) h\Big(\frac{b^2 - b_0^2}{b_0^2}\Big)\frac{db^2}{b_0^2} \times g(\nu - \nu_0)\, d\nu\,, \tag{10}$$

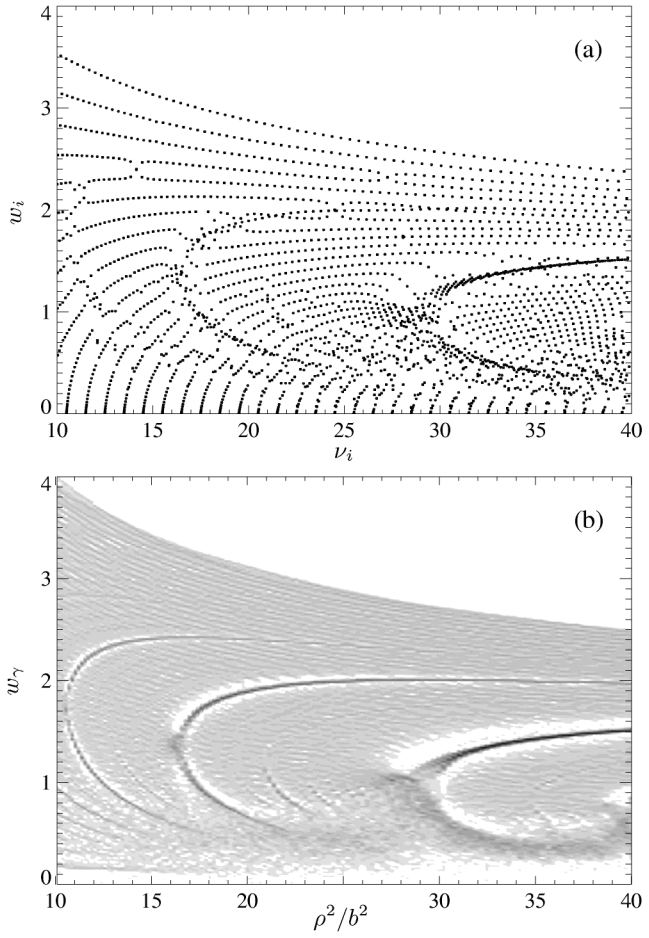

FIG. 2. (a) Weighted edge state spectrum compared to (b) the phase space distribution of the classical weights for the interior ellipse. The quantum weights (8) segregate the edge states from the bulk and mimic the structures in the distribution of classical weights (9).

defined for fixed reference energy ν_0 and magnetic length b_0. The functions g and h are normalized Gaussians. g regularizes the integral over the pair distribution $\tilde{d}^{(\mathrm{int})}_{\mathrm{edge}}(\nu; b^2)\tilde{d}^{(\mathrm{ext})}_{\mathrm{edge}}(\nu; b^2)$ and h restricts the b^2 integration to a range where a linear expansion of $\nu_n(b^2)$ near b_0^2 is allowed. Then, $C(\nu_0)$ can be written as a double sum over the interior and the exterior edge spectra obtained at one fixed value b_0 of the magnetic length:

$$C(\nu_0) = \sum_{i,j=1}^{\infty} \frac{w_i w_j'}{w_i + w_j'}\, g\Bigg(\frac{\frac{\nu_i - \nu_0}{w_i} - \frac{\nu_0 - \nu_j'}{w_j'}}{\frac{1}{w_i} + \frac{1}{w_j'}}\Bigg) h\Big(\frac{\nu_i - \nu_j'}{w_i + w_j'}\Big) - C_{\mathrm{background}}\,. \tag{11}$$

The primes label the exterior energies and weights. Since the width of g is taken small, $\sigma_g \ll (\overline{d}_{\mathrm{edge}})^{-1}$, only those pairs of interior and exterior energies contribute whose distances to ν_0 *scaled* by the respective quantum weights are approximately equal. The prefactor in (11) ensures that only pairs of edge energies contribute.

When evaluating $C(\nu_0)$ semiclassically, the integration over b^2 in (10) selects those pairs in the sum over interior and exterior orbits γ and γ' which satisfy $w_\gamma \mathcal{L}_\gamma \simeq w_{\gamma'}\mathcal{L}_{\gamma'}$, a relation fulfilled by the *dual* pairs. Restricting the summation to the latter is tantamount to the "diagonal approximation" [18]. The actions complement each other to $2\pi\nu_0 n_\gamma$ where $2\pi\nu_0$ is the scaled action of a cyclotron orbit and n_γ is the number of reflections. We obtain

$$C(\nu_0) = \sum_{n=n_{\min}}^{\infty} f(n)\hat{g}(n)\cos[2\pi n(\nu_0 - \tfrac{1}{2})] \tag{12}$$

with $\hat{g}$ the Fourier transform of g. Here, $n_{\min}$ is the minimal number of reflections needed for a periodic orbit at given ρ and

$$f(n) = \frac{2}{\pi}\sum_{\gamma\in\Gamma_n} a_\gamma^2 w_\gamma^2 \tag{13}$$

a sum over the set Γ_n of dual orbits with n reflections. It involves the classical weights w_γ (9) and the stability amplitudes a_γ [14] of the unweighted spectral density.

From its semiclassical form (12) the correlator is expected to be appreciably different from zero only at energies where the cosine terms are stationary. Hence, $C(\nu_0)$ must exhibit peaks at $\nu_0 = N + \frac{1}{2}$. Its Fourier transform, on the other hand, should be peaked at the integer values starting from $n_{\min}$.

A numerical verification of these predictions is presented in Fig. 3 for the spectra of the ellipse billiard at $b_0 = 0.1$ [9]. In Fig. 3(a) one observes that the cross correlation function is strongly fluctuating but displays pronounced spikes at the expected energies. They are a clear signature of a correlation between the interior and exterior edge states. The peaks in the Fourier transform, Fig. 3(b), are positioned at integer values of t which start at $n_{\min} = 4$, as expected for the (desymmetrized) ellipse. They expose clearly the classical duality as the origin of the cross correlations.

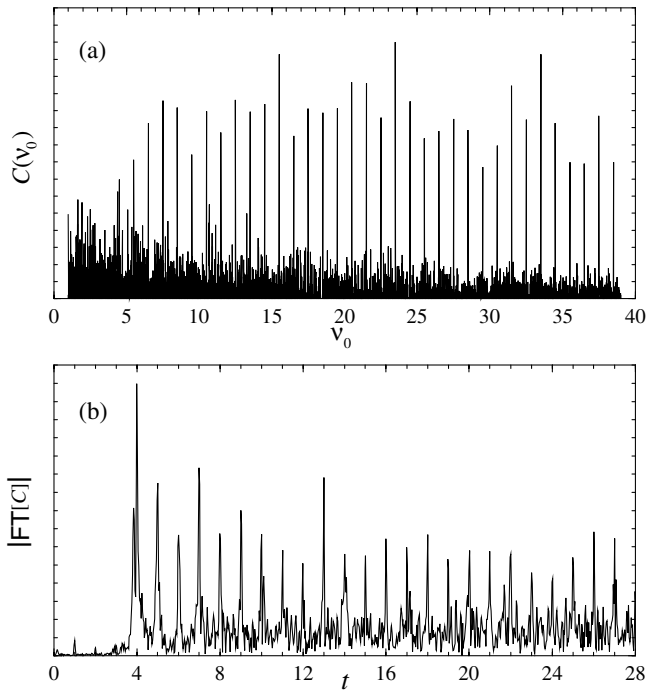

FIG. 3. (a) Cross-correlation function (11) for the elliptic billiard. The pronounced spikes indicate the existence of pairwise correlations between interior and exterior edge states. (b) Fourier transform of (11). The peaks at integers starting from 4 prove the classical origin of the correlations ($\sigma_g = 0.001$).

As a test we restricted the sum (11) to pairs taken from the two different symmetry classes of the ellipse. This erases the peaks in $C(\nu_0)$ as one expects semiclassically. On the other hand, removing the bulk states by imposing a threshold on the w_i does not change Fig. 3.

The spikes of $C(\nu_0)$ imply the existence of a *pairwise* relation between the interior and exterior edge states: For each correlated pair of interior and exterior edge energies, ν_i and ν_j', there exists a Landau level $\nu_0 = N + \frac{1}{2}$ such that the distances—scaled individually by the quantum weights w_i and w_j'—are approximately equal,

$$\frac{\nu_i - (N - \frac{1}{2})}{w_i} \simeq \frac{(N - \frac{1}{2}) - \nu_j'}{w_j'}. \qquad (14)$$

This follows immediately from (11), where we took the width of g to be small on the quantum scale. The relation (14) shows that the interior and exterior edge spectra are intimately connected in the semiclassical limit. Note in particular the vital role played by the quantum weights (8) without which the correlations would not be observable.

Equation (14) allows us to spot single pairs of correlated states in the spectrum, and Fig. 1(b) gives an outstanding illustration (with the shade proportional to $|\psi|^2$): The wave functions are clearly localized along the stable dual periodic orbits drawn in Fig. 1(a). Although the respective energies are separated by 20 mean edge state spacings, $\nu_i = 31.425\,54$, $\nu_j' = 31.616\,96$, the difference between the two sides of (14) with $w_i = 17.23/\nu_i$, $w_j' = 26.76/\nu_j'$, and $N = 31$ is approximately one-tenth of the mean level spacing scaled by the mean weight. The correlation of pairs of *chaotic* wave functions cannot be verified as easily by visual inspection, but they exhibit a large overlap of their normal derivatives at the boundary.

Let us finally emphasize that the correlations between edge states of the interior and the exterior do not permit us to derive one spectrum from the other—even in the semiclassical limit—since the respective Landau level N and the complementary quantum weights are not known *a priori*. Nonetheless, they quantify a deep interconnection of the spectra which is generated by the classical duality, as initially conjectured from Fig. 1. Similar cross correlations may be expected between nonmagnetic systems as well, e.g., in complementary billiards on the sphere.

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