Optomechanical sensing of spontaneous wave-function collapse – Supplementary Material

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1 Diffusion predicted by CSL and DP

Here we derive explicitly the momentum diffusion rates D_{CSL} and D_{DP} predicted by the CSL and the DP model, which result in Equations (1) and (11) in the main text.

The CSL master equation for a system of N masses m_n with position operators r_n reads in a first-quantization picture as [1, 2]

$$\mathcal{L}_{\mathrm{CSL}}\rho = \frac{\lambda_{\mathrm{CSL}}}{\pi^{3/2} r_{\mathrm{CSL}}^3 \mathrm{amu}^2} \int \mathrm{d}^3 s \left[\mathsf{m}(\boldsymbol{s})\rho\mathsf{m}(\boldsymbol{s}) - \frac{1}{2}\left\{\rho, \mathsf{m}^2(\boldsymbol{s})\right\}\right], \quad \mathsf{m}(\boldsymbol{s}) = \sum_n m_n \exp\left[-\frac{(\boldsymbol{s}-\mathsf{r}_n)^2}{2r_{\mathrm{CSL}}^2}\right]. \tag{S1}$$

The operator $\mathsf{m}(\boldsymbol{s})$ describes a Gaussian-averaged mass density of the N-particle system. In the case of a rigid compound system, the position operator of each particle, $\mathsf{r}_n = \mathsf{r} + \boldsymbol{r}_n^{(0)} + \Delta\mathsf{r}_n$, can be expressed in terms of the center-of-mass position operator r of the whole object and $N-1$ relative coordinates. The latter describe the confined motion of the rigidly bound constituents around their equilibrium configuration $\boldsymbol{r}_n^{(0)}$ in the center-of-mass system. This motion can be safely neglected, because it is bound to scales well below the CSL localization length $r_{\mathrm{CSL}} = 100\,\mathrm{nm}$ (see also [1], Sect. 8.2). We may then write

$$\mathsf{m}(\boldsymbol{s}) \approx \sum_n m_n \exp\left[-\frac{\left(\boldsymbol{s}-\mathsf{r}-\boldsymbol{r}_n^{(0)}\right)^2}{2r_{\mathrm{CSL}}^2}\right] = \frac{r_{\mathrm{CSL}}^3}{(2\pi)^{3/2}} \int \mathrm{d}^3 k\, \exp\left[-\frac{r_{\mathrm{CSL}}^2 k^2}{2}\right] e^{i\boldsymbol{k}\cdot(\boldsymbol{s}-\mathsf{r})} \underbrace{\sum_n m_n e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_n^{(0)}}}_{\equiv\tilde{\varrho}(\boldsymbol{k})}, \tag{S2}$$

introducing the Fourier transform $\tilde{\varrho}(\boldsymbol{k})$ of the object's mass density $\varrho(\boldsymbol{r}) = \sum_n m_n \delta\left(\boldsymbol{r}-\boldsymbol{r}_n^{(0)}\right)$. For the CSL model, the latter can be replaced by the homogeneous mass density of the object, as explained in the main text. The CSL master equation (S1) now acts on the center-of-mass state of motion,

$$\mathcal{L}_{\mathrm{CSL}}\rho = \frac{r_{\mathrm{CSL}}^3 \lambda_{\mathrm{CSL}}}{\pi^{3/2}\mathrm{amu}^2} \int \mathrm{d}^3 k\, e^{-r_{\mathrm{CSL}}^2 k^2} |\tilde{\varrho}(\boldsymbol{k})|^2 \left(e^{i\boldsymbol{k}\cdot\mathsf{r}}\rho e^{-i\boldsymbol{k}\cdot\mathsf{r}} - \rho\right). \tag{S3}$$

The exponential operators can be expanded to lowest order in the one-dimensional center-of-mass coordinate x in the present case, where the center-of-mass motion is restricted to linear oscillations over amplitudes along the x-axis much smaller than r_{CSL}. This results in the diffusive form $\mathcal{L}_{\mathrm{CSL}}\rho \approx -D_{\mathrm{CSL}}\left[\mathsf{x},[\mathsf{x},\rho]\right]/\hbar^2$, with the diffusion rate $D_{\mathrm{CSL}} = \lambda_{\mathrm{CSL}}(\hbar/r_{\mathrm{CSL}})^2\alpha$ used in the main text, see Equation (1).

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The DP result (11) is obtained analogously after rewriting the DP master equation [3] for an object of mass density $\varrho(\boldsymbol{r})$ by means of a Fourier transform,

$$\mathcal{L}_{\rm DP}\rho = -\frac{G}{2\hbar}\int \frac{\mathrm{d}^3 s_1 \mathrm{d}^3 s_2}{|\boldsymbol{s}_1 - \boldsymbol{s}_2|}\left[\varrho\left(\boldsymbol{s}_1 - \mathbf{r}\right), \left[\varrho\left(\boldsymbol{s}_2 - \mathbf{r}\right), \rho\right]\right] = \frac{G}{2\pi^2\hbar}\int \frac{\mathrm{d}^3 k}{k^2}\,|\tilde{\varrho}(\boldsymbol{k})|^2 \left(e^{i\boldsymbol{k}\cdot\mathbf{r}}\rho e^{-i\boldsymbol{k}\cdot\mathbf{r}} - \rho\right). \tag{S4}$$

Note that the Fourier transform of the Coulomb-like term is taken to be the usual $4\pi/k^2$. The diffusion rate $D_{\rm DP}$ describes the average growth rate in the second moment of the momentum induced by the above generator $\mathcal{L}_{\rm DP}$. If we are only interested in the one-dimensional motion along the x-axis, then $D_{\rm DP} = \mathrm{tr}\left(\mathsf{p}_x^2 \mathcal{L}_{\rm DP}\rho\right)$ leads to the expression (11) in the main text.

2 CSL diffusion for cuboids, spheres and discs

Here we present the exact expressions for the geometry factors, Eq. (1) in the main text, of homogeneous rigid bodies of mass m and mean density $\varrho = m/V$. For cuboids of volume $V_{\rm cuboid} = b_x b_y b_z$, discs of volume $V_{\rm disc} = \pi R^2 d$, and spheres of volume $V_{\rm sphere} = 4\pi R^3/3$, the difference lies in the mass density function and its Fourier transform,

$$\tilde{\varrho}_{\rm cuboid}(k_x, k_y, k_z) = m\,\mathrm{sinc}\left(\frac{k_x b_x}{2}\right)\mathrm{sinc}\left(\frac{k_y b_y}{2}\right)\mathrm{sinc}\left(\frac{k_z b_z}{2}\right), \tag{S5}$$

$$\tilde{\varrho}_{\rm disc}(k_x, \boldsymbol{k}_\perp) = \frac{2m}{k_\perp R} J_1(k_\perp R)\mathrm{sinc}\left(\frac{k_x d}{2}\right), \tag{S6}$$

$$\tilde{\varrho}_{\rm sphere}(\boldsymbol{k}) = 3m\frac{\sin kR - kR\cos kR}{(kR)^3}. \tag{S7}$$

Here, J_1 denotes a Bessel function. After plugging these expressions into the geometry factor (1) in the main text, which determines the momentum diffusion rate for the one-dimensional motion along the x-axis, a tedious but straightforward calculation yields

$$\alpha_{\rm cuboid} = \left(\frac{m}{\rm amu}\right)^2 \Gamma_1\left(\frac{b_y}{\sqrt{2} r_{\rm CSL}}\right)\Gamma_1\left(\frac{b_z}{\sqrt{2} r_{\rm CSL}}\right)\left[1 - e^{-b_x^2/4r_{\rm CSL}^2}\right]\frac{2r_{\rm CSL}^2}{b_x^2}, \tag{S8}$$

$$\alpha_{\rm disc} = \left(\frac{m}{\rm amu}\right)^2 \Gamma_\perp\left(\frac{R}{\sqrt{2} r_{\rm CSL}}\right)\left[1 - e^{-d^2/4r_{\rm CSL}^2}\right]\frac{2r_{\rm CSL}^2}{d^2}, \tag{S9}$$

$$\alpha_{\rm sphere} = \left(\frac{m}{\rm amu}\right)^2 \left[e^{-R^2/r_{\rm CSL}^2} - 1 + \frac{R^2}{2r_{\rm CSL}^2}\left(e^{-R^2/r_{\rm CSL}^2} + 1\right)\right]\frac{6r_{\rm CSL}^6}{R^6}, \tag{S10}$$

with the abbreviations

$$\Gamma_1(\xi) = \frac{2}{\xi^2}\left[e^{-\xi^2/2} - 1 + \sqrt{\frac{\pi}{2}}\xi\,\mathrm{erf}\left(\frac{\xi}{\sqrt{2}}\right)\right], \qquad \Gamma_\perp(\xi) = \frac{2}{\xi^2}\left\{1 - e^{-\xi^2}\left[I_0(\xi^2) + I_1(\xi^2)\right]\right\}. \tag{S11}$$

The terms $I_{0,1}$ denote modified Bessel functions. Equations (2) and (3) in the main text are obtained by expanding the exact geometry factors in $R/r_{\rm CSL}$, $b/r_{\rm CSL}$, and $d/r_{\rm CSL}$ asymptotically.

3 DP diffusion for cubic crystal lattices

Here we calculate the DP diffusion rate, Eq. (11) in the main text, for macroscopic solids consisting of a cubic and monoatomic crystal lattice. For simplicity, we neglect the small electron mass and assume that the nuclear mass $m_{\rm A}$ at each lattice point is on average distributed evenly according to the Gaussian mass density distribution $\varrho_{\rm A}(\boldsymbol{r}) = m_{\rm A}\exp(-r^2/2\sigma_{\rm DP}^2)/(2\pi\sigma_{\rm DP}^2)^{3/2}$, with $\tilde{\varrho}_{\rm A}(\boldsymbol{k}) = m_{\rm A}\exp(-\sigma_{\rm DP}^2 k^2/2)$ its Fourier transform.

The spread σ_{DP} is assumed to be much smaller than the lattice constant a, so that the average mass densities of neighboring lattice points do not overlap.

The total mass density of the object and its Fourier transform can now be written as

$$\begin{aligned} \varrho(\boldsymbol{r}) &= \chi(\boldsymbol{r}) \sum_{j,n,\ell=-\infty}^{\infty} \varrho_{\mathrm{A}}(x - ja, y - na, z - \ell a), \\ \tilde{\varrho}(\boldsymbol{k}) &= \frac{1}{a^3} \sum_{j,n,\ell} \tilde{\varrho}_{\mathrm{A}} \left(\frac{2\pi j}{a}, \frac{2\pi n}{a}, \frac{2\pi \ell}{a} \right) \tilde{\chi} \left(k_x - \frac{2\pi j}{a}, k_y - \frac{2\pi n}{a}, k_z - \frac{2\pi \ell}{a} \right) \end{aligned} \tag{S12}$$

where $\chi(\boldsymbol{r})$ denotes the characteristic function of the given body shape; it is unity for all points inside the body volume and zero elsewhere (i.e. χ is proportional to the homogeneous mass density employed in the CSL case). Its Fourier transform is denoted by $\tilde{\chi}$. The function $\chi(\boldsymbol{r})$ varies on essentially macroscopic scales, whereas the lattice sum is a sharply peaked periodic function oscillating on the microscopic scales σ_{DP}, a. Given the macroscopic volume of the object, $V^{1/3} \gg a \gg \sigma_{\mathrm{DP}}$, the Fourier transform $\tilde{\chi}$ of the characteristic function has a width of the order of $V^{-1/3}$, much smaller than $2\pi/a$. Hence, we may reduce the double summation to a single sum when taking the absolute square of $\tilde{\varrho}(\boldsymbol{k})$ and write

$$|\tilde{\varrho}(\boldsymbol{k})|^2 \approx \frac{1}{a^6} \sum_{j,n,\ell} \left| \tilde{\varrho}_{\mathrm{A}} \left(\boldsymbol{G}_{jn\ell} \right) \tilde{\chi} \left(\boldsymbol{k} - \boldsymbol{G}_{jn\ell} \right) \right|^2, \tag{S13}$$

with $\boldsymbol{G}_{jn\ell} = 2\pi (j, n, \ell)/a$ a reciprocal lattice vector. Plugging this into the DP diffusion rate (11) and exploiting once again the sharply peaked nature of $\tilde{\chi}$, we arrive at

$$D_{\mathrm{DP}} = \frac{G\hbar}{2\pi^2 a^6} \sum_{jn\ell} \left| \tilde{\varrho}_{\mathrm{A}} \left(\boldsymbol{G}_{jn\ell} \right) \right|^2 \int \mathrm{d}^3 k \, \frac{k_x^2}{\boldsymbol{k}^2} \left| \tilde{\chi} \left(\boldsymbol{k} - \boldsymbol{G}_{jn\ell} \right) \right|^2 \approx \frac{G\hbar}{2\pi^2 a^6} \sum_{jn\ell} \left| \tilde{\varrho}_{\mathrm{A}} \left(\boldsymbol{G}_{jn\ell} \right) \right|^2 \frac{j^2}{j^2 + n^2 + \ell^2} \int \mathrm{d}^3 k \, |\tilde{\chi}(\boldsymbol{k})|^2 . \tag{S14}$$

The latter integral can be evaluated using $\chi^2 = \chi$, that is, $\int \mathrm{d}^3 k \, |\tilde{\chi}(\boldsymbol{k})|^2 = (2\pi)^3 \int \mathrm{d}^3 r \, \chi^2(\boldsymbol{r}) = (2\pi)^3 V$. Moreover, the function $\tilde{\varrho}_{\mathrm{A}}(\boldsymbol{G}_{jn\ell})$ extends over many reciprocal lattice vectors, since $1/\sigma_{\mathrm{DP}} \gg 2\pi/a$. This allows us to approximate the lattice sum by an integral,

$$D_{\mathrm{DP}} \approx \frac{4\pi G\hbar V}{a^6} \left(\frac{a}{2\pi} \right)^3 \int \mathrm{d}^3 q \, \frac{q_x^2}{\boldsymbol{q}^2} |\tilde{\varrho}_{\mathrm{A}}(\boldsymbol{q})|^2 = \frac{G\hbar m_{\mathrm{A}}^2 V}{6\pi^2 a^3} \int \mathrm{d}^3 q \, e^{-\sigma_{\mathrm{DP}}^2 q^2} = \frac{G\hbar m_{\mathrm{A}}^2 V}{6\sqrt{\pi} a^3 \sigma_{\mathrm{DP}}^3} \tag{S15}$$

Noting that $m = m_{\mathrm{A}} V/a^3$ is the total mass of the object and $\varrho = m_{\mathrm{A}}/a^3$ its mean density, we arrive at the result (12) given in the main text.

References

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