Supplemental material to 'Force-gradient sensing and entanglement via feedback cooling of interacting nanoparticles'

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SI. DERIVATION OF THE FEEDBACK MASTER EQUATION

This section derives the feedback quantum master equation for a dielectric sub-wavelength particle with volume V and susceptibility tensor $\chi(\Omega)$, depending on the particle orientation Ω. The particle is illuminated by a monomchromatic optical field of wavenumber k, which can always be expressed in the spectral representation [1]

$$\mathbf{E}(\mathbf{r}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^2\mathbf{n}\, \mathbf{E}(\mathbf{n}) e^{ik\mathbf{n}\cdot\mathbf{r}}, \tag{S1}$$

with $\mathbf{n}\cdot\mathbf{E}(\mathbf{n}) = 0$ due to transversality. For positions far away from the origin, one can expand (S1) in powers of $1/r$ as

$$\mathbf{E}(r\mathbf{n}) = \frac{1}{(2\pi)^{1/2} ikr} \left[\mathbf{E}(\mathbf{n})e^{ikr} - \mathbf{E}(-\mathbf{n})e^{-ikr}\right], \tag{S2}$$

neglecting orders of $O(1/r^2)$. Therefore we can identify an outgoing part of the incident wave, proportional to the spectral representation at a given travelling direction $\mathbf{n}$, and an ingoing part, proportional to the spectral representation at $-\mathbf{n}$.

The particle dynamics are determined by the optical potential $V(\mathbf{r},\Omega) = -\varepsilon_0 V \mathbf{E}^*(\mathbf{r})\cdot\chi(\Omega)\mathbf{E}(\mathbf{r})/4$, with $\mathbf{r}$ the particle center of mass, as well as by the scattering Lindblad operators [2]

$$L^{\rm sc}_{\mathbf{n}p} = \sqrt{\frac{\varepsilon_0 k^3}{2\hbar}} \frac{V}{4\pi} \mathbf{t}^*_{\mathbf{n}p}\cdot\chi(\Omega)\mathbf{E}(\mathbf{r})e^{-ik\mathbf{n}\cdot\mathbf{r}}. \tag{S3}$$

Here, $\mathbf{n}$ describes the direction of the scattered photon and $\mathbf{t}_{\mathbf{n}p}$, $p = 1,2$, the two orthogonal polarisation directions. The resulting quantum master equation reads [2]

$$\begin{aligned}\partial_t\rho &= \mathcal{L}_0\rho - \frac{i}{\hbar}[V_{\rm opt},\rho] \\ &+ \int d^2\mathbf{n} \sum_{p=1,2}\left(L^{\rm sc}_{\mathbf{n}p}\rho(L^{\rm sc}_{\mathbf{n}p})^\dagger - \frac{1}{2}\{(L^{\rm sc}_{\mathbf{n}p})^\dagger L^{\rm sc}_{\mathbf{n}p},\rho\}\right). \end{aligned}\tag{S4}$$

The laser-free particle dynamics, described by $\mathcal{L}_0$, includes gas diffusion.

We now follow the reasoning from [3] to derive a master equation conditioned on the outcome of a homodyne measurement on the scattered light. For this we first reformulate equation (S4) to account for the fact that the total field measured at a detector is given not only by the scattered field but also by the outgoing part of $\mathbf{E}(\mathbf{r})$. Defining the operators

$$L_{\mathbf{n}p} = \sqrt{\frac{\varepsilon_0}{4\pi\hbar k^3}} \frac{1}{i} \mathbf{t}^*_{\mathbf{n}p}\cdot\mathbf{E}(\mathbf{n}) + L^{\rm sc}_{\mathbf{n}p}, \tag{S5}$$

allows rewriting (S4) as

$$\partial_t\rho = \mathcal{L}_0\rho + \int d^2\mathbf{n}\sum_{p=1,2}\left(L_{\mathbf{n}p}\rho L^\dagger_{\mathbf{n}p} - \frac{1}{2}\{L^\dagger_{\mathbf{n}p}L_{\mathbf{n}p},\rho\}\right). \tag{S6}$$

The laser-induced dynamics are now fully encoded in the new Lindblad operators, which account for the superposition of the scattered field and the outgoing part of the incident field. Note that

$$\int d^2\mathbf{n}\sum_{p=1,2} L^\dagger_{\mathbf{n}p}L_{\mathbf{n}p} = \frac{P}{\hbar ck}, \tag{S7}$$

with P the total incident field power.

We next show that the Lindblad operators can be understood as the measurement operators of a detector counting all outgoing photons. In order to model a homodyne measurement at the shot noise limit, all observable photon scattering directions are individually superposed with classical local oscillator fields $\beta_{\mathbf{n}p}$ and then summed over all accessible scattering directions [3]. The latter are determined by the solid angle Ω_0 of the light collecting objective. The detector quantum efficiency is denoted as η_0 and we include dark counts with rate r, so that the measurement signal is a Poisson process $dN(t)$ with mean

$$\begin{aligned}\mathbb{E}[dN(t)] &= r dt \\ &+\eta_0 \int_{\Omega_0} d^2\mathbf{n}\sum_{p=1,2}\mathbb{E}[\langle(\beta^*_{\mathbf{n}p} + L^\dagger_{\mathbf{n}p})(\beta_{\mathbf{n}p}+L_{\mathbf{n}p})\rangle]dt.\end{aligned}\tag{S8}$$

From this, we can infer the measurement operators to be $\beta_{\mathbf{n}p} + L_{\mathbf{n}p}$ [3].

Unravelling the master equation (S6) with respect to the signal (S8) requires a few steps. We first rewrite (S6) as

$$\partial_t\rho = \mathcal{L}_0\rho + \int d^2\mathbf{n}\sum_{p=1,2}\left[L_{\mathbf{n}p}\rho L^\dagger_{\mathbf{n}p} - \frac{1}{2}\{L^\dagger_{\mathbf{n}p}L_{\mathbf{n}p},\rho\}\right] - \eta_0\int_{\Omega_0} d^2\mathbf{n}\sum_{p=1,2}\left[L_{\mathbf{n}p}\rho L^\dagger_{\mathbf{n}p} - \frac{1}{2}\{L^\dagger_{\mathbf{n}p}L_{\mathbf{n}p},\rho\}\right]$$
$$+\eta_0\int_{\Omega_0} d^2\mathbf{n}\sum_{p=1,2}\left[(\beta_{\mathbf{n}p}+L_{\mathbf{n}p})\rho(\beta^*_{\mathbf{n}p}+L^\dagger_{\mathbf{n}p}) - \frac{1}{2}\{(\beta^*_{\mathbf{n}p}+L^\dagger_{\mathbf{n}p})(\beta_{\mathbf{n}p}+L_{\mathbf{n}p}),\rho\}\right] - \frac{\eta_0}{2}\int_{\Omega_0} d^2\mathbf{n}\sum_{p=1,2}\left[\beta^*_{\mathbf{n}p}L_{\mathbf{n}p} - \beta_{\mathbf{n}p}L^\dagger_{\mathbf{n}p},\rho\right], \tag{S9}$$

by using that the last term in the first line cancels the second line. Next, the first term in the second line is stochastically unravelled by using (S8),

$$d\rho = \mathcal{L}_0\rho dt + \int d^2\mathbf{n}\sum_{p=1,2}\left[L_{\mathbf{n}p}\rho L^\dagger_{\mathbf{n}p} - \frac{1}{2}\{L^\dagger_{\mathbf{n}p}L_{\mathbf{n}p},\rho\}\right]dt - \eta_0\int_{\Omega_0} d^2\mathbf{n}\sum_{p=1,2}\left[L_{\mathbf{n}p}\rho L^\dagger_{\mathbf{n}p} - \frac{1}{2}\{L^\dagger_{\mathbf{n}p}L_{\mathbf{n}p},\rho\}\right]dt$$
$$+\eta_0\int_{\Omega_0} d^2\mathbf{n}\sum_{p=1,2}\left[\langle(\beta^*_{\mathbf{n}p}+L^\dagger_{\mathbf{n}p})(\beta_{\mathbf{n}p}+L_{\mathbf{n}p})\rangle\rho - \frac{1}{2}\{(\beta_{\mathbf{n}p}+L_{\mathbf{n}p})^\dagger(\beta_{\mathbf{n}p}+L_{\mathbf{n}p}),\rho\}\right]dt$$
$$+\left[\frac{r\rho + \eta_0\int_{\Omega_0} d^2\mathbf{n}\sum_{p=1,2}(\beta_{\mathbf{n}p}+L_{\mathbf{n}p})\rho(\beta^*_{\mathbf{n}p}+L^\dagger_{\mathbf{n}p})}{r+\eta_0\int_{\Omega_0} d^2\mathbf{n}\sum_{p=1,2}\langle(\beta^*_{\mathbf{n}p}+L^\dagger_{\mathbf{n}p})(\beta_{\mathbf{n}p}+L_{\mathbf{n}p})\rangle} - \rho\right]dN(t) - \frac{\eta_0}{2}\int_{\Omega_0} d^2\mathbf{n}\sum_{p=1,2}\left[\beta^*_{\mathbf{n}p}L_{\mathbf{n}p} - \beta_{\mathbf{n}p}L^\dagger_{\mathbf{n}p},\rho\right]dt. \tag{S10}$$

This equation describes the collapse of the quantum state due to the detection or not-detection of a photon. A detection event ($dN = 1$) transforms the state into a mixture of the pre-detection state (if the photon was a dark count) and a superposition state consisting of the pre-detection state (if the photon was from the local oscillator) and the state for a photon leaving the system, averaged over all detection directions Ω_0.

For homodyne detection, the field of the local oscillator is much greater than the field to be detected. In the formal limit $\beta_{\mathbf{n}p} \to \infty$ the number of photons arriving at the detector in a finite time interval tends to infinity, while the state transformation of a single detection event vanishes. The usual approach to describe this situation slices the time axis into intervals of finite duration Δt, which must fulfill two conditions [3]: First, the slices have to be long enough so that the number of photons arriving at the detector during this period clearly exceeds unity. Second, the slices have to be sufficiently short so that the quantum state changes only weakly during Δt. The number of photons $\Delta N(t)$ detected during Δt is then approximately given by a Gaussian distribution, whose variance equals its mean (S8). One can thus write [3]

$$\Delta N(t) \approx \left[r+\eta_0\int_{\Omega_0} d^2\mathbf{n}\sum_{p=1,2}\langle(\beta^*_{\mathbf{n}p}+L^\dagger_{\mathbf{n}p})(\beta_{\mathbf{n}p}+L_{\mathbf{n}p})\rangle\right]\Delta t + \sqrt{r+\eta_0\int_{\Omega_0} d^2\mathbf{n}\sum_{p=1,2}\langle(\beta^*_{\mathbf{n}p}+L^\dagger_{\mathbf{n}p})(\beta_{\mathbf{n}p}+L_{\mathbf{n}p})\rangle}\,\Delta W(t) \tag{S11}$$

where $\Delta W(t)$ is a Gaussian distributed random number with zero mean and variance Δt.

Since Δt is much smaller than the evolution time of the master equation, we can replace dN in equation (S10) by ΔN. Keeping only terms to leading order in $\beta^{-1}_{\mathbf{n}p}$, performing the limit $\Delta t \to dt$, and splitting the Lindblad operators according to equation (S5) yields

$$d\rho = \mathcal{L}_0\rho dt - \frac{i}{\hbar}[V_{\rm opt}(\mathbf{r}),\rho]dt + \int d^2\mathbf{n}\sum_{p=1,2}\left[L^{\rm sc}_{\mathbf{n}p}\rho(L^{\rm sc}_{\mathbf{n}p})^\dagger - \frac{1}{2}\{(L^{\rm sc}_{\mathbf{n}p})^\dagger L^{\rm sc}_{\mathbf{n}p},\rho\}\right]dt$$
$$+\frac{\eta_0}{\sqrt{r+\eta_0\int_{\Omega_0} d^2\mathbf{n}\sum_{p=1,2}|\beta_{\mathbf{n}p}|^2}}\left[\int_{\Omega_0} d^2\mathbf{n}\sum_{p=1,2}\beta^*_{\mathbf{n}p}L^{\rm sc}_{\mathbf{n}p}\rho + \beta_{\mathbf{n}p}\rho(L^{\rm sc}_{\mathbf{n}p})^\dagger - \langle\beta^*_{\mathbf{n}p}L^{\rm sc}_{\mathbf{n}p} + \beta_{\mathbf{n}p}(L^{\rm sc}_{\mathbf{n}p})^\dagger\rangle\rho\right]dW(t), \tag{S12}$$

with Wiener increment $dW(t)$. This quantum master equation describes the motion of an arbitrarily shaped nanoparticle subject to continuous homodyning. The homodyne photon flux $I_{\rm hom}$ is obtained by substracting the constant contributions due to the local oscillator, the dark counts, and the incident field $\mathbf{E}$ from the photon count,

$$I_{\rm hom}(t)dt = \eta_0\int_{\Omega_0} d^2\mathbf{n}\sum_{p=1,2}\langle\beta^*_{\mathbf{n}p}L^{\rm sc}_{\mathbf{n}p} + \beta_{\mathbf{n}p}(L^{\rm sc}_{\mathbf{n}p})^\dagger\rangle dt + \sqrt{r+\eta_0\int_{\Omega_0} d^2\mathbf{n}\sum_{p=1,2}|\beta_{\mathbf{n}p}|^2}\,dW(t). \tag{S13}$$

The signal is therefore proportional to the scattering field of the particle and is fluctuating due to the photon shot noise of the local oscillator and the dark counts.

As a final step, we now use that the particle is spherical, $\chi(\Omega) = \chi_{\mathrm{e}}\mathbb{1} = 3(\varepsilon_{\mathrm{r}} - 1)\mathbb{1}/(\varepsilon_{\mathrm{r}} + 2)$, and that the incident field is a tweezer trap of the form

$$\mathbf{E}(\mathbf{r}) = \frac{E_0}{1 + ix/x_{\mathrm{R}}} \exp\left[-\frac{y^2 + z^2}{w^2\,(1 + ix/x_{\mathrm{R}})}\right] e^{i(kx+\varphi_{\mathrm{t}})}\mathbf{e}_z, \tag{S14}$$

with waist w, Rayleigh range $x_{\mathrm{R}} = kw^2/2$, polarisation $\mathbf{e}_z$, amplitude E_0, and tweezer phase φ_{t}. Linearizing Eq. (S12) around the tweezer focus and tracing out orientational degrees of freedom shows that the deterministic part of the linearised master equation consists of a harmonic trap for all center-of-mass degrees of freedom and Rayleigh scattering diffusion for all components of $\mathbf{r}$ [2]. The stochastic term in (S12) can be treated by expanding $L^{\mathrm{sc}}_{\mathbf{n}p}$ to the first order in the particle coordinates

$$L^{\mathrm{sc}}_{\mathbf{n}p} \approx \sqrt{\frac{\varepsilon_0 k^3}{2\hbar}} \frac{V\chi_{\mathrm{e}}E_0}{4\pi} \mathbf{t}^*_{\mathbf{n}p} \cdot \mathbf{e}_z \left\{1 + i\left[\left(k - \frac{1}{x_{\mathrm{R}}}\right)x - k\mathbf{n}\cdot\mathbf{r}\right]\right\} e^{i\varphi_{\mathrm{t}}}. \tag{S15}$$

The constant term cancels and thus does not appear in equation (S12).

Finally, we assume that the detector solid angle Ω_0 and the local oscillator $\beta_{\mathbf{n}p}$ are chosen such that the measured signal is approximately independent of y and z. Tracing out y and z from the master equation and choosing $\mathcal{L}_0$ as the unitary evolution due to the kinetic energy plus a contribution due to gas scattering with diffusion constant D_{g} finally leads to the feedback master equation for the coordinate x along the optical axis,

$$d\rho = -\frac{i}{\hbar}[H,\rho]dt - \left(\frac{D_{\mathrm{g}}}{\hbar^2} + \frac{1}{8L^2}\right)[x,[x,\rho]]dt + \frac{\sqrt{\eta}}{2L}\left(e^{i\varphi}x\rho + e^{-i\varphi}\rho x - 2\cos\varphi\langle x\rangle\rho\right)dW. \tag{S16}$$

Here, we defined the detection phase

$$\varphi = \varphi_{\mathrm{t}} + \arg\left[\int_{\Omega_0} d^2\mathbf{n} \sum_{p=1,2} i(\mathbf{t}^*_{\mathbf{n}p}\cdot\mathbf{e}_z)\beta^*_{\mathbf{n}p}\left(k - \frac{1}{x_{\mathrm{R}}} - k\mathbf{n}\cdot\mathbf{e}_x\right)\right] \tag{S17}$$

and the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega_0^2}{2}x^2 - F(t)x, \tag{S18}$$

with external force $F(t)$ and trapping frequency $\omega_0 = \sqrt{\chi_{\mathrm{e}}kP/\pi c\varrho x_{\mathrm{R}}^3}$. We define the detection efficiency as

$$\eta = \frac{\eta_0^2\left|\int_{\Omega_0} d^2\mathbf{n}\sum_{p=1,2}(\mathbf{t}^*_{\mathbf{n}p}\cdot\mathbf{e}_z)\beta^*_{\mathbf{n}p}\left(k - \frac{1}{x_{\mathrm{R}}} - kn_x\right)\right|^2}{\left[\int d^2\mathbf{n}\sum_{p=1,2}|\mathbf{t}_{\mathbf{n}p}\cdot\mathbf{e}_z|^2\left(k - \frac{1}{x_{\mathrm{R}}} - kn_x\right)^2\right]\left[r + \eta_0\int_{\Omega_0} d^2\mathbf{n}\sum_{p=1,2}|\beta_{\mathbf{n}p}|^2\right]}, \tag{S19}$$

which, following Ref. [4], can be understood as a product of several efficiencies, $\eta = \eta_0\eta_{\mathrm{inel}}\eta_{\mathrm{ov}}\eta_{\mathrm{dc}}$. Specifically, η_{inel} denotes the fraction of inelastically scattered photons arriving at the detector

$$\eta_{\mathrm{inel}} = \frac{\int_{\Omega_0} d^2\mathbf{n}\sum_{p=1,2}|\mathbf{t}_{\mathbf{n}p}\cdot\mathbf{e}_z|^2\left(k - \frac{1}{x_{\mathrm{R}}} - kn_x\right)^2}{\int d^2\mathbf{n}\sum_{p=1,2}|\mathbf{t}_{\mathbf{n}p}\cdot\mathbf{e}_z|^2\left(k - \frac{1}{x_{\mathrm{R}}} - kn_x\right)^2}, \tag{S20}$$

while η_{ov} describes the mode overlap of the detectable inelastically scattered photons with the local oscillator

$$\eta_{\mathrm{ov}} = \frac{\left|\int_{\Omega_0} d^2\mathbf{n}\sum_{p=1,2}(\mathbf{t}^*_{\mathbf{n}p}\cdot\mathbf{e}_z)\beta^*_{\mathbf{n}p}\left(k - \frac{1}{x_{\mathrm{R}}} - kn_x\right)\right|^2}{\left[\int_{\Omega_0} d^2\mathbf{n}\sum_{p=1,2}|\mathbf{t}_{\mathbf{n}p}\cdot\mathbf{e}_z|^2\left(k - \frac{1}{x_{\mathrm{R}}} - kn_x\right)^2\right]\left[\int_{\Omega_0} d^2\mathbf{n}\sum_{p=1,2}|\beta_{\mathbf{n}p}|^2\right]}. \tag{S21}$$

Likewise, η_{dc} is the mean probability for a detected photon to originate from the local oscillator and not from the dark counts

$$\eta_{\mathrm{dc}} = \frac{\eta_0\int_{\Omega_0} d^2\mathbf{n}\sum_{p=1,2}|\beta_{\mathbf{n}p}|^2}{r + \eta_0\int_{\Omega_0} d^2\mathbf{n}\sum_{p=1,2}|\beta_{\mathbf{n}p}|^2}. \tag{S22}$$

We note that most of the weight of $\eta_{\rm inel}$, and therefore most of the particle information, is encoded in the backscattered light, for which n_x is negative.

The homodyne photon flux can be written as $I_{\rm hom}(t)dt = I_1 dt + I_2 dy(t)$ with the increment $dy = \langle x\rangle dt\cos\varphi + LdW/\sqrt{\eta}$ and

$$I_1 = \eta_0 \int_{\Omega_0} d^2\mathbf{n} \sum_{p=1,2} \sqrt{\frac{\varepsilon_0 k^3}{2\hbar}} \frac{V\chi_{\rm e}E_0}{4\pi} \beta^*_{\mathbf{n}p}\mathbf{t}^*_{\mathbf{n}p}\cdot\mathbf{e}_z e^{i\varphi_{\rm t}} + \text{c.c.} \tag{S23a}$$

$$I_2 = \frac{\sqrt{\eta}}{L}\sqrt{r + \eta_0 \int_{\Omega_0} d^2\mathbf{n} \sum_{p=1,2} |\beta_{\mathbf{n}p}|^2}. \tag{S23b}$$

Substracting the constant photon flux and renormalizing the homodyne signal with I_2 thus measures the particle position through $dy(t)$.

Extending this derivation to the case of two particles and two far-detuned tweezers with two individual detectors and transforming to mechanical normal modes, leads us to the feedback master equation (4) for $\varphi = 0$. The master equation (S16) can also be obtained by performing an infinite series of infinitely weak Gaussian measurements on the position operator [5]; it is a well-known description applicable to several contemporary experiments [4, 6].

SII. EQUILIBRIUM POSITIONS IN THE PRESENCE OF COULOMB ATTRACTION

The electrostatic interaction not only couples the motion of the two particles along all coordinates, but also displaces their equilibrium positions along their connecting axis (y). As the motion along the beam polarisation (z) does not influence the particle motion along the optical axis (x), we ignore the former in the following considerations.

The potential energy of the y- and x-coordinates of both particles is given by the sum of the optical potentials of both particles in their respective beams (contributions of the other tweezer can be neglected for all parameters considered in our work) plus the electrostatic interaction. Additionally, a constant and homogeneous electrostatic field $E_{\rm c}$ can be applied along the y-axis. The coordinates $\mathbf{r}_j = (x_j, y_j, z_j)$ of particle $j = 1, 2$ refer to the respective beam focus. Then, the potential reads

$$V(x_1, x_2, y_1, y_2) = -\frac{\varepsilon_0\chi_{\rm e}V}{4}|\mathbf{E}(\mathbf{r}_1)|^2 - \frac{\varepsilon_0\chi_{\rm e}V}{4}|\mathbf{E}(\mathbf{r}_2)|^2 + \frac{Q_1Q_2}{4\pi\varepsilon_0|d\mathbf{e}_y + \mathbf{r}_2 - \mathbf{r}_1|} - Q_1E_{\rm c}y_1 - Q_2E_{\rm c}y_2. \tag{S24}$$

For repulsive interaction, $Q_1Q_2 > 0$, we choose $E_{\rm c} = 0$. As the motion along the optical axis is only stable for relatively low coupling rates, $g/\omega_0 > -1/4$, the potential can safely be expanded around the two tweezer foci. For attractive interaction and larger coupling rates, however, the constant displacement of the equilibrium positions along y may change the effective trapping frequencies and recoil rates, and may even destabilise the traps. This may be compensated by choosing $E_{\rm c}$ as

$$E_{\rm c} = \frac{Q_1Q_2}{(Q_1 - Q_2)2\pi\varepsilon_0 d^2}. \tag{S25}$$

The resulting potential energy of both particles exhibits extrema at $x_{0,1} = x_{0,2} = 0$ and $y_{0,1} = y_{0,2}$, provided a solution $y_{0,1}$ of the following transcendental equation exists,

$$\frac{y_{0,1}}{w}\exp\left(-2\frac{y_{0,1}^2}{w^2}\right) = -\frac{wd}{x_{\rm R}^2}\left(\frac{g}{\omega_0}\right)_{y=0}\frac{Q_1+Q_2}{Q_1-Q_2}, \tag{S26}$$

where $(g/\omega_0)_{y=0}$ is the ratio of the bare coupling rate and trapping frequency as introduced in the main text. A solution exists if the right-hand side is smaller than the maximum of the left-hand side, $1/2e^{1/2}$, which implies $y_{0,1} < w/2$. The effective trapping frequencies along the optical axis are then given by

$$\omega_0^2 = \frac{\chi_{\rm e}kP}{\pi c\varrho x_{\rm R}^3}\left(1 - 2\frac{y_{0,1}^2}{w^2}\right)\exp\left(-2\frac{y_{0,1}^2}{w^2}\right), \tag{S27}$$

and are thus always reduced in comparison to the bare trapping frequency., Likewise, the recoil heating rate is reduced by a factor of $\exp\left(-2y_{0,1}^2/w^2\right)$. The coupling rate remains unchanged, apart from its dependence on ω_0. We note that for an equal number of absolute charges on the particles, $|Q_1| = |Q_2|$, the displacement along y can be fully canceled. Equal charges would however prevent feedback cooling of the sum mode via electric fields and thus require optical cold damping cooling [7, 8].

The trapping frequency of the center of mass of the particles along y is given by

$$\omega_{y,+}^2 = \frac{4\chi_\mathrm{e}P}{\pi c\varrho w^4}\left(1-4\frac{y_{01}^2}{w^2}\right)\exp\left(-2\frac{y_{01}^2}{w^2}\right), \tag{S28}$$

and the frequency of the difference mode by $\omega_{y,-}^2 = \omega_{y,+}^2 - 4g\omega_0$. Note that the trapping of the difference mode along y can become unstable for sufficiently strong attractive interaction ($\omega_{y,+}^2 < 4g\omega_0$), even if $y_{0,1}$ exists. This can however be circumvented with feedback, by superimposing E_c with an additional field proportional to $y_2 - y_1$ to add an additional restoring force.

SIII. MOMENT EQUATIONS OF MOTION AND STATIONARY CONDITIONAL COVARIANCE

From Eq. (4) we can derive the equations of motion for the moments of $x_\pm$ and $p_\pm$. Denoting the conditional covariance of operators A and B as $C_{AB} = \frac{1}{2}\langle AB + BA\rangle - \langle A\rangle\langle B\rangle$ and the variance of A as $V_A = C_{AA}$, the first moments can be shown to evolve as

$$d\langle x_\pm\rangle = \frac{\langle p_\pm\rangle}{m}dt + \frac{\sqrt{\eta}}{L}V_{x_\pm}dW_\pm + \frac{\sqrt{\eta}}{L}C_{x_+x_-}dW_\mp \tag{S29a}$$

$$d\langle p_\pm\rangle = -m\omega_\pm^2\langle x_\pm\rangle dt + F_\pm(t) + K_\pm(t) + \frac{\sqrt{\eta}}{L}C_{x_\pm p_\pm}dW_\pm + \frac{\sqrt{\eta}}{L}C_{x_\mp p_\pm}dW_\mp. \tag{S29b}$$

Here, $\eta = \eta_\mathrm{in} + \eta_\mathrm{out}$ and $dW_\pm = (\sqrt{\eta_\mathrm{in}}dW_{\mathrm{in},\pm} + \sqrt{\eta_\mathrm{out}}dW_{\mathrm{out},\pm})/\sqrt{\eta}$.

Likewise, the dynamics of the (co-)variances are given by the following set of equations of motion.

$$\begin{aligned}
dV_{x_\pm} =& \frac{2C_{x_\pm p_\pm}}{m}dt - \frac{\eta}{L^2}V_{x_\pm}^2 dt - \frac{\eta}{L^2}C_{x_+x_-}^2 dt + \frac{\sqrt{\eta}}{L}(\langle x_\pm^3\rangle - 3V_{x_\pm}\langle x_\pm\rangle - \langle x_\pm\rangle^3)dW_\pm \\
&+ \frac{\sqrt{\eta}}{L}(\langle x_\pm^2 x_\mp\rangle - V_{x_\pm}\langle x_\mp\rangle - 2C_{x_+x_-}\langle x_\pm\rangle - \langle x_\pm\rangle^2)dW_\mp
\end{aligned} \tag{S30a}$$

$$\begin{aligned}
dC_{x_+x_-} =& \frac{C_{x_+p_-} + C_{x_-p_+}}{m}dt - \frac{\eta}{L^2}(V_{x_+} + V_{x_-})C_{x_+x_-}dt \\
&+ \frac{\sqrt{\eta}}{L}(\langle x_+^2 x_-\rangle - 2C_{x_+x_-}\langle x_+\rangle - V_{x_+}\langle x_-\rangle - \langle x_+\rangle^2\langle x_-\rangle)dW_+ \\
&+ \frac{\sqrt{\eta}}{L}(\langle x_-^2 x_+\rangle - 2C_{x_+x_-}\langle x_-\rangle - V_{x_-}\langle x_+\rangle - \langle x_-\rangle^2\langle x_+\rangle)dW_-
\end{aligned} \tag{S30b}$$

$$\begin{aligned}
dC_{x_\pm p_\pm} =& \frac{V_{p_\pm}}{m}dt - m\omega_\pm^2 V_{x_\pm}dt - \frac{\eta}{L^2}V_{x_\pm}C_{x_\pm p_\pm}dt - \frac{\eta}{L^2}C_{x_+x_-}C_{x_\mp p_\pm}dt \\
&+ \frac{\sqrt{\eta}}{L}(\langle x_\pm p_\pm x_\pm\rangle - 2C_{x_\pm p_\pm}\langle x_\pm\rangle - V_{x_\pm}\langle p_\pm\rangle - \langle x_\pm\rangle^2\langle p_\pm\rangle)dW_\pm \\
&+ \frac{\sqrt{\eta}}{L}\left(\frac{1}{2}\langle x_\pm p_\pm x_\mp + x_\mp p_\pm x_\pm\rangle - C_{x_\pm p_\pm}\langle x_\mp\rangle - C_{x_+x_-}\langle p_\pm\rangle - C_{x_\mp p_\pm}\langle x_\pm\rangle - \langle x_+\rangle\langle x_-\rangle\langle p_\pm\rangle\right)dW_\mp
\end{aligned} \tag{S30c}$$

$$\begin{aligned}
dC_{x_\pm p_\mp} =& \frac{C_{p_+p_-}}{m}dt - m\omega_\mp^2 C_{x_+x_-}dt - \frac{\eta}{L^2}V_{x_\pm}C_{x_\pm p_\mp}dt - \frac{\eta}{L^2}C_{x_+x_-}C_{x_\mp p_\pm}dt \\
&+ \frac{\sqrt{\eta}}{L}(\langle x_\pm^2 p_\mp\rangle - 2C_{x_\pm p_\mp}\langle x_\pm\rangle - V_{x_\pm}^2\langle p_\mp\rangle - \langle x_\pm\rangle^2\langle p_\mp\rangle)dW_\pm \\
&+ \frac{\sqrt{\eta}}{L}\left(\frac{1}{2}\langle x_\mp p_\mp x_\pm + x_\pm p_\mp x_\mp\rangle - C_{x_\pm p_\mp}\langle x_\mp\rangle - C_{x_\mp p_\mp}\langle x_\pm\rangle - C_{x_+x_-}\langle p_\mp\rangle - \langle x_+\rangle\langle x_-\rangle\langle p_\mp\rangle\right)dW_\mp
\end{aligned} \tag{S30d}$$

$$\begin{aligned}
dV_{p_\pm} =& -2m\omega_\pm^2 C_{x_\pm p_\pm}dt + \left(\frac{\hbar^2}{4L^2} + 2D_\mathrm{g}\right)dt - \frac{\eta}{L^2}C_{x_\pm p_\pm}^2 dt - \frac{\eta}{L^2}C_{x_\mp p_\pm}^2 dt \\
&+ \frac{\sqrt{\eta}}{L}(\langle p_\pm x_\pm p_\pm\rangle - V_{p_\pm}\langle x_\pm\rangle - 2C_{x_\pm p_\pm}\langle p_\pm\rangle - \langle p_\pm\rangle^2\langle x_\pm\rangle)dW_\pm \\
&+ \frac{\sqrt{\eta}}{L}(\langle p_\pm^2 x_\mp\rangle - V_{p_\pm}\langle x_\mp\rangle - 2C_{x_\mp p_\pm}\langle p_\pm\rangle - \langle p_\pm\rangle^2\langle x_\mp\rangle)dW_\mp
\end{aligned} \tag{S30e}$$

$$dC_{p_+p_-} = -m(\omega_+^2 C_{x_+p_-} + \omega_-^2 C_{x_-p_+})dt - \frac{\eta}{L^2}C_{x_+p_+}C_{x_+p_-}dt - \frac{\eta}{L^2}C_{x_-p_-}C_{x_-p_+}dt$$

$$+\frac{\sqrt{\eta}}{L}\left(\frac{1}{2}\langle p_-x_+p_+ + p_+x_+p_-\rangle - C_{p_+p_-}\langle x_+\rangle - C_{x_+p_+}\langle p_-\rangle - C_{x_+p_-}\langle p_+\rangle - \langle p_+\rangle\langle p_-\rangle\langle x_+\rangle\right)dW_+$$
$$+\frac{\sqrt{\eta}}{L}\left(\frac{1}{2}\langle p_+x_-p_- + p_-x_-p_+\rangle - C_{p_+p_-}\langle x_-\rangle - C_{x_-p_-}\langle p_+\rangle - C_{x_-p_+}\langle p_-\rangle - \langle p_+\rangle\langle p_-\rangle\langle x_-\rangle\right)dW_-. \quad \text{(S30f)}$$

In general, neither the first and second moments nor the subsystems $s = \pm$ decouple from each other since all equations of motion are non-linearly coupled. Moreover, the equations of motion are not closed because third moments appear in the stochastic part of the (co-)variances. While all moments are driven by all measurement noises $dW_{r\pm}$, the (co-)variance equations of motion do not depend on the feedback and external forces $F_\pm$ and $K_\pm$, which has also been noticed for a single particle [4, 6, 9].

The equations (S29) and (S30) cannot be solved in general. However, the problem simplifies greatly for Gaussian states, because they remain Gaussian for all times under equation (S16). To see this, consider an infinitesimal time step, which can be written as $\rho(t+dt) \propto \int_{-\infty}^{\infty} du \exp[-u^2/2]\mathcal{W}(u)\rho(t)\mathcal{W}^\dagger(u)$, where

$$\mathcal{W}(u) = \exp\left[-\frac{i}{\hbar}H_u dt - \eta\frac{(x\cos\varphi - dy(t)/dt)^2}{4L^2}dt\right], \quad \text{(S31)}$$

with

$$H_u = H + \frac{\hbar\eta}{4L^2}\cos\varphi\sin\varphi\, x^2 - \hbar\left(\frac{\sqrt{\eta}}{2L}\sin\varphi\frac{dW(t)}{dt} + u\sqrt{\frac{2D_{\rm g}}{\hbar^2} + \frac{1-\eta}{4L^2}}dt^{-1/2} + \frac{\eta}{2L^2}\cos\varphi\sin\varphi\,\langle x\rangle\right)x. \quad \text{(S32)}$$

As long as H is at most quadratic in all position and momentum operators, the master equation thus conserves the initial state's gaussianity.

Gaussian states fulfill the following relations for their third moments

$$\langle : uvw : \rangle = C_{uv}\langle w\rangle + C_{uw}\langle v\rangle + C_{vw}\langle u\rangle + \langle u\rangle\langle v\rangle\langle w\rangle, \quad \text{(S33)}$$

with u, v and w position or momentum operators and $: uvw :$ denoting their Weyl ordering. This can be shown with the help of the Wigner representation of a Gaussian state by using that the Weyl symbol of a Weyl-ordered operator product is the product of the individual Weyl symbols. The above relation then follows from a classical calculation, using that the Wigner function is a Gaussian.

Using Eq. (S33) we see that all stochastic terms in the dynamics of the second moments (S30) vanish, giving rise to a set of coupled and nonlinear, but closed and deterministic differential equations. After a transient time, the (co-)variances settle at the stationary solutions

$$V_{x_s} = \frac{\sqrt{2}L^2\omega_s}{\eta}\zeta_s \quad \text{(S34a)}$$
$$C_{x_sp_s} = \frac{mL^2\omega_s^2}{\eta}\zeta_s^2 \quad \text{(S34b)}$$
$$V_{p_s} = m^2\omega_s^2 V_{x_s}(1+\zeta_s^2), \quad \text{(S34c)}$$

and $C_{x_+x_-} = C_{p_+p_-} = C_{x_\pm p_\mp} = 0$, where $s \in \{+,-\}$ and

$$\zeta_s^2 = \sqrt{1 + \frac{\eta(\hbar^2 + 8D_{\rm g}L^2)}{4m^2L^4\omega_s^4}} - 1. \quad \text{(S35)}$$

This implies that the conditional sum and difference modes become uncorrelated. The steady state of the conditional covariance matrix is only a property of the measurement setting and not of the external forces or of the applied feedback.

SIV. MEASUREMENT SIGNALS

Inserting the stationary covariance into Eq. (S29) demonstrates that the sum and difference motion is only driven by the sum and difference noises $dW_{r\pm}$, respectively.

Inserting the cold-damping feedback force (5), the equations of motion for the first moments (S29) can be solved in the frequency domain. Denoting all frequency-dependent quantities with square brackets, we define the Fourier transform by $A[\omega] = \int_{-\infty}^{\infty} dt\, e^{i\omega t} A(t)/\sqrt{2\pi}$. The stationary first moments are then given by

$$\langle x_s\rangle[\omega] = \chi_s[\omega]\left[\left(\frac{C_{x_sp_s}}{m} - i\omega V_{x_s}\right)\frac{\sqrt{\eta}}{L}\xi_s[\omega] - \frac{\gamma_s L}{\sqrt{\eta_{\rm in}}}\sqrt{2\pi}H_s[\omega]\xi_{{\rm in},s}[\omega] + \frac{K_s[\omega]}{m}\right], \tag{S36a}$$

$$\langle p_s\rangle[\omega] = -m\chi_s[\omega]\left[\left(i\omega\frac{C_{x_sp_s}}{m} + \omega_s^2 V_{x_s} + \gamma_s\sqrt{2\pi}H_s[\omega]V_{x_s}\right)\frac{\sqrt{\eta}}{L}\xi_s[\omega] - i\omega\frac{\gamma_s L}{\sqrt{\eta_{\rm in}}}\sqrt{2\pi}H_s[\omega]\xi_{{\rm in},s} + i\omega\frac{K_s[\omega]}{m}\right], \tag{S36b}$$

with the mechanical susceptibilities $\chi_s[\omega] = (\omega_s^2 - \omega^2 + \sqrt{2\pi}\gamma_s H_s[\omega])^{-1}$ and the white noises $\xi[\omega]$, as labeled by different subscripts. The latter are the Fourier transforms of the respective dW/dt, so that $\mathbb{E}[\xi[\omega]] = 0$ and $\mathbb{E}[\xi[\omega']^*\xi[\omega]] = \delta(\omega-\omega')$. From these relations one can directly calculate all power spectral densities $S_{AB}[\omega] = \int_{-\infty}^{\infty} d\omega' \mathbb{E}[\langle A\rangle[\omega]^*\langle B\rangle[\omega']]/2\pi$.

At this point we note that the feedback filter function is causal, $H_s(t<0) = 0$, and real-valued, implying that the Fourier transform $H_s[\omega]$ has non-vanishing real and imaginary parts. Specifically, the reality of $H_s(t)$ implies $H_s[\omega] = H_s^*[-\omega]$, so that we can write $H_s[\omega] = (g_s[\omega] - i\omega f_s[\omega])/\sqrt{2\pi}$ with real and symmetric functions $g_s[\omega]$ and $f_s[\omega]$. For the filter function to generate cold damping with rate γ_s, we require $g_s[\omega_s] = 0$ and $f_s[\omega_s] = 1$, with both functions approximately constant for a spectral width of at least γ_s around ω_s. We note that in realistic situations, the filter $H_s(t)$ is always compact, so that $H_s[\omega]$ decays as $1/\omega$ for large ω, implying that $f_s[\omega]$ decays as $1/\omega^2$.

First, we calculate the PSDs of the in-loop measurement signals, $dy_{{\rm in},s}(t)/dt$. They are given by

$$S_{{\rm in},s}[\omega] = \frac{L^2}{2\pi\eta_{\rm in}} + \frac{|\chi_s[\omega]|^2}{2\pi}\left[\frac{\hbar^2}{4m^2L^2} + \frac{2D_{\rm g}}{m^2} - 2\pi\frac{L^2}{\eta_{\rm in}}\gamma_s^2|H_s[\omega]|^2 - 2(\omega_s^2-\omega^2)\frac{L^2}{\eta_{\rm in}}\gamma_s g_s[\omega]\right] + |\chi_s[\omega]|^2\frac{S_{K_sK_s}[\omega]}{m^2}, \tag{S37}$$

showing noise squashing [10] for sufficienctly large damping constants. Neglecting $g_s[\omega]$ yields Eq. (6) and (7) in the main text. Likewise, the PSDs of the out-of-loop signals $dy_{{\rm out},s}(t)/dt$ are also given by Eq. (6) and (7).

SV. ENTANGLING TWO PARTICLES BY COULOMB INTERACTION

For Gaussian quantum states the amount of entanglement is encoded in the covariance matrix of the two particles. We set $K_s = 0$ and $\eta_{\rm out} = 0$ in the following, so that $\eta = \eta_{\rm in}$. To get the unconditional covariance matrix, one may calculate the PSDs $S_{x_sx_s}$, $S_{x_sp_s}$ and $S_{p_sp_s}$ from Eqs. (S36) and integrate them over all frequencies yielding $\mathbb{E}[\langle x_s\rangle^2]$, $\mathbb{E}[\langle x_s\rangle\langle p_s\rangle]$ and $\mathbb{E}[\langle p_s\rangle^2]$.

In order to evaluate these integrals, we will restrict ourselves to cases where $f_s[\omega]$ and $g_s[\omega]$ change weakly around ω_s in an interval larger than γ_s. This applies if the filter $H_s[\omega]$ is much broader than the mechanical PSD. Then, we can set $H_s[\omega] \approx H_s[\omega_s]$ for integrating over $S_{x_sx_s}$ and $S_{x_sp_s}$. Note that the ensemble average correlations between sum and difference modes are zero. When integrating over $S_{p_sp_s}$, however, the finite cutoff of $H_s[\omega]$ plays a role due to high-frequency fluctuations emerging from feeding back the filtered measurement noise into the oscillators. This manifests itself in a term of the form $f_s[\omega]^2\omega^4|\chi_s[\omega]|^2$ in the integrand, where $f_s[\omega]$ determines the convergence of the integral, rather than $|\chi_s[\omega]|^2$. Keeping the exact form of g_s and f_s only where they are needed to achieve convergence yields

$$\mathbb{E}[\langle x_s\rangle^2] = \frac{L^2}{2\eta}\gamma_s + \left(\frac{\hbar^2}{8L^2} + D_{\rm g}\right)\frac{1}{m^2\omega_s^2\gamma_s} - V_{x_s} \tag{S38a}$$

$$\mathbb{E}[\langle p_s\rangle^2] = m^2\omega_s^2\left[\mathbb{E}[\langle x_s\rangle^2] + V_{x_s} + \frac{L^2}{\eta}\left(\zeta_s^2\gamma_s + \frac{\Omega_s}{2\omega_s^2}\gamma_s^2\right)\right] - V_{p_s} \tag{S38b}$$

$$\mathbb{E}[\langle x_s\rangle\langle p_s\rangle] = \frac{mL^2\omega_s}{\sqrt{2}\eta}\zeta_s\gamma_s - C_{x_sp_s}, \tag{S38c}$$

with the filter bandwidths Ω_s. The latter satisfy $\Omega_s \gg \gamma_s$ by construction (and would diverge if H_s acted as an exact derivative). The integral is well-defined in all realistic cases due to the $1/\omega^2$-decay of the integrand. Finally adding the conditional covariance matrix to these results leads us to the unconditional covariance matrix

$$\mathbb{E}[\langle x_s^2\rangle] = \frac{L^2}{2\eta}\gamma_s + \left(\frac{\hbar^2}{8L^2} + D_{\rm g}\right)\frac{1}{m^2\omega_s^2\gamma_s} \tag{S39a}$$

$$\mathbb{E}[\langle p_s^2\rangle] = m^2\omega_s^2\left[\mathbb{E}[\langle x_s^2\rangle] + \frac{L^2}{\eta}\left(\zeta_s^2\gamma_s + \frac{\Omega_s}{2\omega_s^2}\gamma_s^2\right)\right] \tag{S39b}$$

$$\mathbb{E}\left[\frac{1}{2}\langle x_sp_s + p_sx_s\rangle\right] = \frac{mL^2\omega_s}{\sqrt{2}\eta}\zeta_s\gamma_s. \tag{S39c}$$

All other second moments vanish.

Note that the equipartition theorem does not hold in general because the stationary state is not thermal. This is due to the fact that the positions, but not the momenta, of the particles are be measured, leading to an additional momentum uncertainty manifesting in the ζ_s-term in the momentum variances. This term vanishes for sufficiently large measurement uncertainties L [6]. In addition, the high-frequency fluctuations in the position measurement signal lead to a contribution of the feedback bandwidth to the momentum variances, which also violates the equipartition theorem.

To quantify the amount of entanglement present in the two-particle stationary state, we calculate its logarithmic negativity. Recalling that the main text introduces dimensionless position and momentum quadratures via $x_s = \sqrt{\hbar/m\omega_0}X_s$, $p_s = \sqrt{\hbar m\omega_0}P_s$ as well as the net heating rate $\Gamma = \Gamma_{\rm sc} + \gamma_{\rm g} k_{\rm B} T_{\rm g}/\hbar\omega_0$ and the effective detection efficiency $\eta_{\rm eff} = \eta_{\rm in}\Gamma_{\rm sc}/\Gamma$, the elements of the dimensionless conditional covariance matrix can be written as

$$V_{X_s} = \frac{\sqrt{\sqrt{\omega_s^4 + 16\eta_{\rm eff}\omega_0^2\Gamma^2} - \omega_s^2}}{4\sqrt{2}\eta_{\rm eff}\Gamma} \tag{S40a}$$

$$V_{P_s} = \frac{\omega_s^2}{\omega_0^2}V_{X_s} + \frac{32\eta_{\rm eff}^2\Gamma^2}{\omega_0^2}V_{X_s}^3 \tag{S40b}$$

$$C_{X_sP_s} = \frac{4\eta_{\rm eff}\Gamma}{\omega_0}V_{X_s}^2. \tag{S40c}$$

Thus, the purity of the conditional state is $\eta_{\rm eff}$, as estimated from Eq. (7) in [11]. The ensemble-averaged second moments (unconditional covariances) are

$$\mathbb{E}[\langle X_s^2\rangle] = \frac{\gamma_s}{16\eta_{\rm eff}\Gamma} + \frac{\omega_0^2\Gamma}{\omega_s^2\gamma_s} \tag{S41a}$$

$$\mathbb{E}[\langle P_s^2\rangle] = \frac{2\sqrt{\omega_s^4 + 16\eta_{\rm eff}\omega_0^2\Gamma^2} - \omega_s^2}{16\eta_{\rm eff}\omega_0^2\Gamma}\gamma_s + \frac{\Gamma}{\gamma_s} + \frac{\Omega_s\gamma_s^2}{16\eta_{\rm eff}\omega_0^2\Gamma} \tag{S41b}$$

$$\mathbb{E}\left[\frac{1}{2}\langle X_sP_s + P_sX_s\rangle\right] = \frac{\sqrt{\sqrt{\omega_s^4 + 16\eta_{\rm eff}\omega_0^2\Gamma^2} - \omega_s^2}}{8\sqrt{2}\eta_{\rm eff}\omega_0\Gamma}\gamma_s. \tag{S41c}$$

For bipartite Gaussian states the logarithmic negativity can be written as $E_N = \max[0, -\log_2(2\min[c_1, c_2])]$, where $c_{1,2}$ are the symplectic eigenvalues of the partially transposed covariance matrix [12]. In our system, where all correlations between sum and difference mode vanish, the symplectic eigenvalues are

$$c_{1,2}^2 = \frac{1}{2}\left(V_{X_+}V_{P_-} + V_{X_-}V_{P_+} - 2C_{X_+P_+}C_{X_-P_-}\right) \pm \sqrt{\frac{1}{4}\left(V_{X_+}V_{P_-} - V_{X_-}V_{P_+}\right)^2 + \left(V_{X_-}C_{X_+P_+} - V_{X_+}C_{X_-P_-}\right)\left(V_{P_-}C_{X_+P_+} - V_{P_+}C_{X_-P_-}\right)}. \tag{S42}$$

First, we calculate the negativity of the conditional state, which acts as a fundamental limit of optimised cooling schemes [4, 6]. We restrict our discussion to weak measurements, where $\Gamma \ll \omega_0$, for simplicity. Then, in the first non-vanishing order of Γ/ω_0, the position variance can be written as

$$V_{X_s} \approx \frac{\omega_0}{2\sqrt{\eta_{\rm eff}}\omega_s} - \frac{\sqrt{\eta_{\rm eff}}\omega_0^3\Gamma^2}{\omega_s^5}. \tag{S43}$$

Inserting it into V_{P_s} and $C_{X_sP_s}$ and calculating the symplectic eigenvalues to the first non-vanishing order in Γ/ω_0 yields the conditional minimum symplectic eigenvalue

$$\min(c_1, c_2) \approx \sqrt{\frac{\omega_<}{4\eta_{\rm eff}\omega_>}}\left[1 + \frac{\eta_{\rm eff}\Gamma^2}{\omega_0^2}h\left(\frac{\omega_0}{\omega_-}\right)\right], \tag{S44}$$

which must fulfill $\min(c_1, c_2) < 1/2$ for the particles to be entangled. Here, we defined $h(s) = |1-s|(s^4 + 2s^3 + 2s + 1)/(1+s)$. The logarithmic negativity is then given by

$$E_N \approx \max\left[0, \frac{1}{2}\log_2\frac{\omega_>\eta_{\rm eff}}{\omega_<} - \frac{\eta_{\rm eff}\Gamma^2}{\ln 2\,\omega_0^2}h\left(\frac{\omega_0}{\omega_-}\right)\right]. \tag{S45}$$

In the limit $\Gamma/\omega_0 \to 0$ the correlator $C_{X_sP_s}$ vanishes and the symplectic eigenvalues take the simple form $\sqrt{V_{X_\pm}V_{P_\mp}}$ [13]. Then, the conditional state of the particles is entangled if the condition $\eta_{\rm eff} > \omega_</\omega_>$ is fulfilled. In Fig. S1 (b)-(e) we show the logarithmic negativity as a function of coupling constant, heating rate, and effective detection efficiency if using the approximation $\sqrt{V_{X_\pm}V_{P_\mp}}$ as symplectic eigenvalues. Comparing it to Fig. 3, we see that the simplified expression predicts weaker entanglement than the exact negativity, but is still a stronger criterion than the Duan inequality.

Second, we will turn to the entanglement of the unconditional state, as achievable with cold-damping feedback. This requires replacing the (co-)variances in Eq. (S42) with the unconditional second moments (S41). Again, we restrict the discussion to $\Gamma \ll \omega_0$ for simplicity. Since the damping constants γ_s can be freely chosen we always take them to maximise the entanglement between the two particles. The dependence of the logarithmic negativity on the two damping constants γ_s, depicted in Fig. 3 (a), shows a strong maximum for specific choices of the γ_s.

We will see that the optimal choice leads to $\gamma_s \propto \Gamma$. Therefore, considering only the first non-vanishing order in Γ/ω_0 leads to Eq. (9), where the term due to the filter

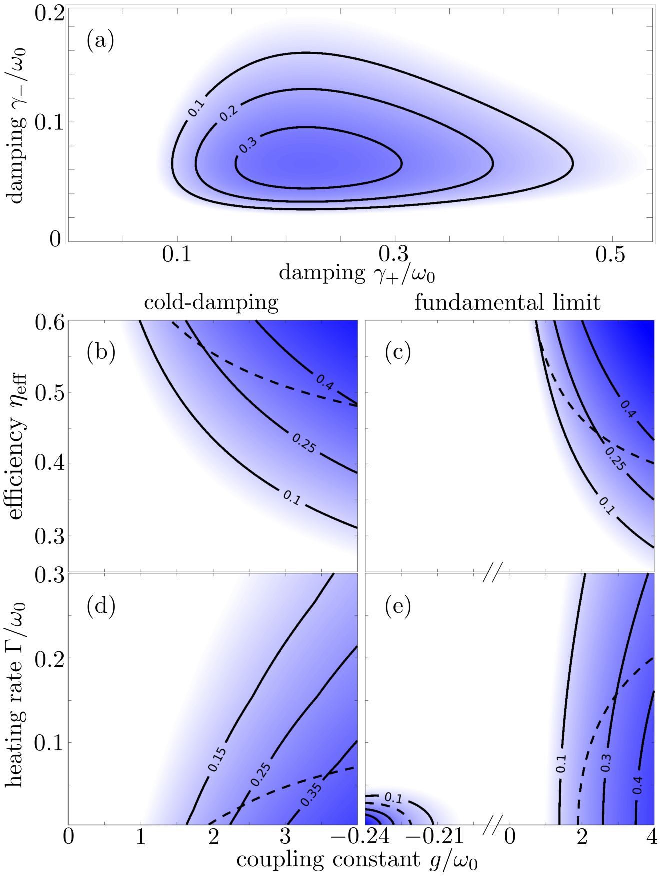

Figure S1. (a) Logarithmic negativity, using the approximation $c_{1,2}^2 \approx \mathbb{E}[\langle X_\pm^2\rangle]\mathbb{E}[\langle P_\mp^2\rangle]$ for the symplectic eigenvalues, of two interacting, feedback-cooled nanoparticles as a function of the damping constants γ_s for an effective detection efficiency of $\eta_{\rm eff} = 0.45$, a net heating rate of $\Gamma = \omega_0/10$ and a coupling constant of $g = 4\omega_0$. (b), (d) Approximated logarithmic negativity of the unconditional state as a function of the coupling constant and of the effective detection efficiency or the net heating rate, respectively. The plots are calculated for $\Omega_s = \omega_0$ and at the optimal choice of the damping rates γ_s. (c), (e) Approximated logarithmic negativity of the conditional state. The dashed black lines indicate the violation of the Duan-criterion. (b), (c) are calculated for $\Gamma = \omega_0/10$, while (d), (e) assume $\eta_{\rm eff} = 0.45$.

width Ω_s is always greater than corrections on the order of Γ^2/ω_0^2 because $\Omega_s \gg \Gamma$.

We note that the correlators $\mathbb{E}\left[\frac{1}{2}\langle X_sP_s + P_sX_s\rangle\right]$ are on the order of Γ/ω_0 and appear quadratically in the logarithmic negativity. They can thus be neglected, so that agin the simplified formula $c_{1,2}^2 \approx \mathbb{E}[\langle X_\pm^2\rangle]\mathbb{E}[\langle P_\mp^2\rangle]$ for the symplectic eigenvalues can be used, see also Fig. S1. Hence, when maximising the negativity we can minimise $\mathbb{E}[\langle X_\pm^2\rangle]$ and $\mathbb{E}[\langle P_\mp^2\rangle]$ individually. To minimise the position variance we choose $\gamma_\pm = 4\Gamma\sqrt{\eta_{\rm eff}}\omega_0/\omega_\pm$, while the momentum fluctuations can be minimised to leading order in Γ/ω_0 with

$$\gamma_\mp \approx 4\Gamma\sqrt{\eta_{\rm eff}}\frac{\omega_0}{\omega_\mp} - \frac{16\eta_{\rm eff}\omega_0^2\Omega_\mp\Gamma^2}{\omega_\mp^4}. \tag{S46}$$

These choices lead to

$$\mathbb{E}[\langle X_\pm^2\rangle] = \frac{\omega_0}{2\sqrt{\eta_{\rm tot}}\omega_\pm} \tag{S47a}$$

$$\mathbb{E}[\langle P_\mp^2\rangle] \approx \frac{\omega_\mp}{2\sqrt{\eta_{\rm eff}}\omega_0} + \frac{\Omega_\mp\Gamma}{\omega_\mp^2}. \tag{S47b}$$

The minimum symplectic eigenvalue then reads

$$\min(c_1, c_2) \approx \sqrt{\frac{\omega_<}{4\eta_{\rm eff}\omega_>}}\left(1 + \sqrt{\eta_{\rm eff}}\frac{\omega_>\Omega_<\Gamma}{\omega_-\omega_<^2}\right), \tag{S48}$$

allowing us to estimate the logarithmic negativity to leading order in Γ/ω_0,

$$E_N \approx \max\left[0, \frac{1}{2}\log_2\left(\eta_{\rm eff}\frac{\omega_>}{\omega_<}\right) - \frac{\sqrt{\eta_{\rm eff}}}{\ln 2}\frac{\omega_>}{\omega_<}\frac{\Omega_<}{\omega_<}\frac{\Gamma}{\omega_-}\right], \tag{S49}$$

where $\omega_< = \min(\omega_+, \omega_-)$ and $\omega_> = \max(\omega_+, \omega_-)$, with $\Omega_<$ the associated filter bandwidth. In the limit of weak coupling, $|g| \ll \omega_0$, and $\Omega_+ \approx \Omega_-$, the entanglement criterion can be simplified to $|g| > 2n_+\omega_0$, where $n_+ = \mathbb{E}[\langle X_+^2 + P_+^2 - 1\rangle]/2$ is the stationary occupation of the sum mode. The resulting logarithmic negativity is given by $E_N \approx \max[0, |g|/\omega_0 - 2n_+]/\ln 2$, resembling the expression for particles coupled to a cold bath [13].

In the limit of $\Gamma/\omega_0 \to 0$, the logarithmic negativities of the conditional and of the unconditional state become identical. The logarithmic negativity when approximating the symplectic eigenvalues with $c_{1,2}^2 \approx \mathbb{E}[\langle X_\pm^2\rangle]\mathbb{E}[\langle P_\mp^2\rangle]$ is depicted in Fig. S1 (a), (b) and (d), where we numerically maximised the approximated negativity (or the violation of the Duan criterion) in (b) and (d) with respect to γ_s. It demonstrates, together with Fig. 3, that one needs larger coupling constants and greater efficiencies to violate the Duan criterion than needed to generate entanglement.

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